A solution set for Chapters 1-3 to *The Theoretical Biologist's Toolbox*

Weclome to this solution set. My name is Erin Middleton. I'll be receiving my BA in Environmental Studies with a minor in Mathmatics at University of California Santa Cruz in June, 2008. I've been working through the *The Theoretical Biologist's Toolbox* with Marc. This set contains solutions to Chapters 1 through 3.

Exercise 1.1 (E)

The two prey diet choice problem- Show that the rate of flow of energy for a forager when they generalize and eat either prey upon encounter is $R_G = \frac{E_1\lambda_1+E_2\lambda_2}{(1+h_1\lambda_1+h_2\lambda_2)}$. T is the total time, H is the handeling time, and S is the time spent searching. E is the energy gain, and λ is the rate that prey are encountered by the predator.

$$\begin{aligned}
T &= S + H_1 + H_2 \\
T &= S + h_1\lambda_1 S + h_2\lambda_2 S \\
T &= S(1 + h_1\lambda_1 + h_2\lambda_2) \\
S &= \frac{T}{1 + h_1\lambda_1 + h_2\lambda_2} \\
R_G &= \frac{E_1\lambda_1 + E_2\lambda_2}{(1 + h_1\lambda_1 + h_2\lambda_2)}
\end{aligned}$$

Exercise 1.2 (E)

Show that $R_s > R_g$ implies that $\lambda_1 > \frac{E_2}{E_1h_2 - E_2h_1}$

Since we know

$$R_s = \frac{E_1\lambda_1}{1 + h_1\lambda_1}$$

and

$$R_g = \frac{E_1\lambda_1 + E_2\lambda_2}{1 + h_1\lambda_1 + h_2\lambda_2}$$

then by our given

$$\frac{E_1\lambda_1}{1 + h_1\lambda_1} > \frac{E_1\lambda_1 + E_2\lambda_2}{1 + h_1\lambda_1 + h_2\lambda_2}$$

Then we solve for λ_1:

$$\begin{aligned}
E_1\lambda_1(1 + h_1\lambda_1 + h_2\lambda_2) &> (E_1\lambda_1 + E_2\lambda_2)(1 + h_1\lambda_1) \\
E_1\lambda_1\lambda_2h_2 - E_2\lambda_1\lambda_2h_1 &> E_2\lambda_2
\end{aligned}$$

$$\lambda_1(E_1\lambda_2 h_2 - E_2\lambda_2 h_1) > E_2\lambda_2$$
$$\lambda_1 > \frac{E_2\lambda_2}{E_1\lambda_2 h_2 - E_2\lambda_2 h_1}$$
$$\lambda_1 > \frac{E_2}{E_1 h_2 - E_2 h_1}$$

Exercise 1.3 (E)

The gain function, G(t), describes the energetic value of food removed by a forager for residence time, t. Two possible forms for the gain function are $G_1(t) = \frac{at}{(b+t)}$ and $G_2(t) = \frac{at^2}{(b+t^2)}$.

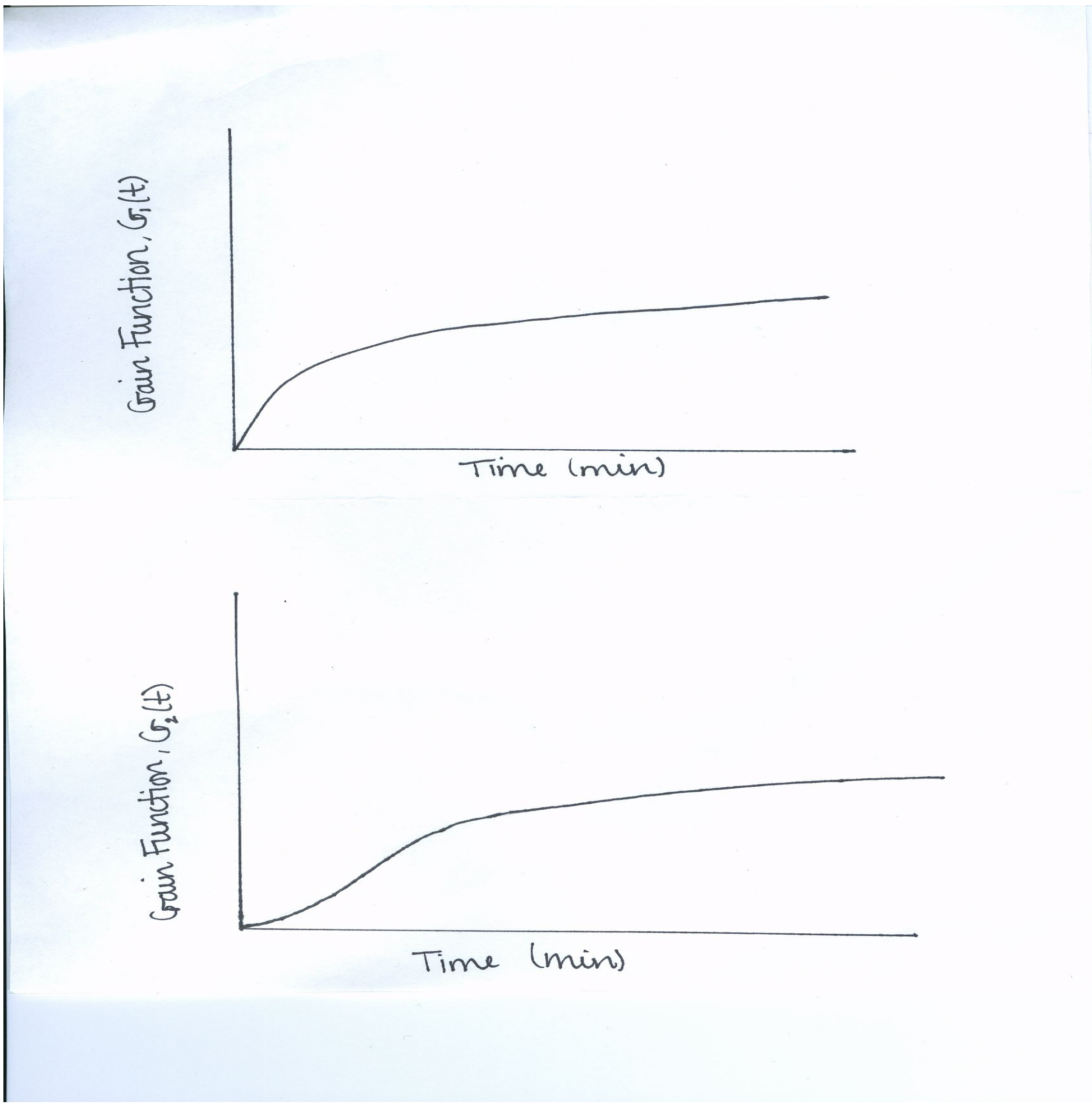


Figure 1. Gain Functions versus Time.

The function $G_2(t)$ describes some kind of initial slowing of the energy gain prior to an inflection point and then experiences a smooth increase until it flattens out implying there is some factor associated with the foraging spot that hinders the organism from gaining energy at a uniform rate.

Variables

Possible values for variables a, b are foraging costs and food abundance in the patches. Note that the same constants (a and b) are used in the expression, but they have different meanings. Since we'll be measuring gain in energy units, *kilocalories*, this implies that b will be in units of time, *minutes* in $G_1(t)$ and *minutes*^{2} in $G_2(t)$.Variable a will be in units of energy, *kilocalories* in $G_1(t)$ and *kilocalories* in $G_2(t)$.

Exercise 1.4 (M)

Show that based on $R(g,x) = \frac{g}{s}, S(x) = \frac{g}{x}(1 - cx^{-a})$ the optimizing value of x is $(c(a+1))^{\frac{1}{a}}$.

$$R(x) = \frac{g}{x}(1 - cx^{-a})$$
$$gR(x) = \frac{1}{x}(1 - cx^{-a})$$
$$gR(x) = x^{-1} - cx^{-a-1}$$

Now we're going to find the derivative of R and set it equal to 0 and solve for x.

$$gR'(x) = -x^{-2} + c(a+1)x^{-a-2} = 0$$
$$\frac{c(a+1)x^{-a-2}}{x^{-a-2}} = \frac{x^{-2}}{x^{-a-2}}$$
$$c(a+1) = x^a$$
$$x = c(a+1)^{\frac{1}{a}}$$

Exercise 1.5- Unbeatable Sex Ratio (M)

$W(r, r^*)$=the fitness depending on upon the sex ratio r that the mutant female uses and the sex ratio r^* that other females use.

E=eggs

r^* =normal individual's fraction of sons

r =mutant individual's fraction of sons

$$W(r, r^*) = E^2(1-r) + E^2[(1-r) + N(1-r^*)\frac{r}{r+Nr^*}]$$

$$\frac{W(r,r^*)}{E^2} = (1-r) + \frac{r(1-r)}{r+Nr^*} + \frac{rN(1-r^*)}{r+Nr^*}$$

Now take the derivative with respect to r and set the equation equal to zero.

$$0 = \frac{d}{dr}(1-r) + \frac{d}{dr}(\frac{r-r^2}{r+Nr^*}) + N(1-r^*)\frac{d}{dr}(\frac{r}{r+Nr^*})$$
$$0 = -1 + \frac{(r+Nr^*)-r}{(r+Nr^*)^2} - \frac{2r(r+Nr^*)-r^2}{(r+Nr^*)^2} + N(1-r^*)\frac{(r+Nr^*)-r}{(r+Nr^*)^2}$$

Now substitute $r = r^*$, since $W(r,r^*)$ is maximized at this point

$$0 = -1 + \frac{(r^*+Nr^*)-r^*}{(r^*+Nr^*)^2} - \frac{2r^*(r^*+Nr^*)-r^{*2}}{(r^*+Nr^*)^2} + N(1-r^*)\frac{r^*+Nr^*-r^*}{(r^*+Nr^*)^2}$$
$$0 = -1 + \frac{Nr}{(r^{*2}+Nr^{*2})^2} - \frac{2r^*(r^*+Nr^*)-r^2}{(r^*+Nr^*)^2} + \frac{N^2r^*(1-r^*)}{(r^*+Nr^*)^2}$$
$$1 = \frac{Nr^* - 2r^{*2} - 2Nr^{*2} + r^{*2} + N^2r^*(1-r^*)}{(r^*+Nr^*)^2}$$
$$(r^*+Nr^*)^2 = Nr^* - 2r^{*2} - 2Nr^{*2} + r^{*2} + N^2r^* - N^2r^{*2}$$
$$r^{*2} + 2Nr^{*2} + N^2r^{*2} = (N^2+N)r^* + (-1-2N-N^2)r^{*2}$$
$$0 = (-2N^2-4N-2)r^{*2} + (N^2+N)r^*$$
$$0 = -2(N+1)^2r^{*2} + (N(N+1))r^* + 0$$

Now apply the quadratic formula:

$$r^* = \frac{-N(N+1)\pm\sqrt{(N(N+1))^2}}{2(-2(N+1)^2)}$$
$$r^* = \frac{-2N(N+1)}{2(-2(N+1)^2)}$$
$$r^* = \frac{N}{2(N+1)}$$

As N $\rightarrow \infty$, $r^* \rightarrow \frac{1}{2}$; this is consistant with Fisherian sex ratios. Refer to book, page 12, on discussions of interpreting the limit as N $\rightarrow 0$.

Exercise 2.1 (E)
Check that the units of q,k, and asymptotic size are correct.

The units of asymptotic size, $\frac{dL}{dt}$:

Equation 2.10 gives us $\frac{dW}{dt} = 3\rho L^2 \frac{dL}{dt}$
Since the units of $\frac{dW}{dt}$ are $\frac{kg}{day}$ then $3\rho L^2 \frac{d}{dt}L$ has the same units.
$\rho = \frac{kg}{cm^3}$
$L^2 = cm^2$
so $\frac{dL}{dt}$ has units $\frac{cm}{day}$.
From $\frac{dL}{dt} = q - kl$ we know q has units $\frac{cm}{day}$.
From $q = \frac{\sigma}{3\rho}$ we can verify the units of q.
Since $\sigma = \frac{kg}{day \cdot cm^2}$ and $\rho = \frac{kg}{cm^3}$, then $q = \frac{kg}{day \cdot cm^2}\frac{cm^3}{kg} = \frac{cm}{day}$.

Similarly with k:
From $\frac{dL}{dt} = q - kL$, kL must be in units $\frac{cm}{day}$.
Thus, k has the units $\frac{1}{day}$.
From $k = \frac{c}{3\rho}$, c has units $\frac{kg}{day \cdot cm}$, and ρ has the units $\frac{kg}{cm^3}$.
Thus, k has the units $\frac{1}{day}$.

For asymptotic size, $L_\infty = \frac{\sigma}{c}$, L_∞ must have the units $\frac{kg}{day \cdot cm^2}\frac{day \cdot cm^3}{kg}$ which simplifies to cm. From our second equation $\frac{dL}{dt} = k(L_\infty - L)$, L is in cm, thus L_∞ is in cm as well.

Exercise 2.2 (M/H)
Show that the solution of Equation 1 with $L(0) = L_0$ is $L(t) = L_\infty(1 - \exp(-kt)) + L_0 \exp(-kt)$.

$$\frac{dL(t)}{dt} = \frac{c}{3\rho}(\frac{\sigma}{c} - L(t)) = k(L_\infty - L(t)) \tag{1}$$

Separate the variables, $L(t)$ and t.

$$\frac{dL(t)}{dt} = k(L_\infty - L(t))$$
$$\frac{dL}{(L_\infty - L)} = kdt$$

Now integrate:

$$\int \frac{dL}{L_\infty - L} = k \int dt$$

Use the following substitution:

$$\begin{aligned}
u &= L_\infty - L \\
du &= -dL \\
f'(u) &= \frac{1}{u} \\
f(u) &= \ln u
\end{aligned}$$

This yields:

$$\begin{aligned}
-\ln(L_\infty - L(t)) &= kt + C \\
\ln(L_\infty - L(t)) &= -kt + C \\
\exp(\ln(L_\infty - L(t))) &= \exp(-kt + C) \\
L_\infty - L(t) &= C \exp(-kt) \\
L(t) &= L_\infty - C \exp(-kt) \qquad (2)
\end{aligned}$$

Initial condition $L(0) = L_0$, $t = 0$

$$\begin{aligned}
L(t) = L_\infty - C \exp(k(0)) &= L_0 \\
C &= L_\infty - L_0
\end{aligned}$$

substitute back into (2).

$$\begin{aligned}
L(t) &= L_\infty - (L_\infty - L_0) \exp(-kt) \\
L(t) &= L_\infty - L_\infty \exp(-kt) + L_0 \exp(-kt) \\
L(t) &= L_\infty(1 - \exp(-kt)) + L_0 \exp(-kt)
\end{aligned}$$

Exercise 2.3 (M)
The variables f and b are parameters, $L(t)$ is length with respect to t, m is the rate of perdation L_0 is the initial time, k is growth, $L_\infty = \frac{\sigma}{c}$.

$$L(t) = L_0 \exp(-kt) + L_\infty(1 - \exp(-kt))$$
$$F(t) = \exp(-mt) f L(t)^b$$

Since f is a constant and we're taking the derivative of $F(t)$ then we can set $f = 1$ and divide through by it.
Note: 'for many organisms, initial size is so small relative to asymptotic size that we can ignore initial size in our manipulations'(p.26). Therefore

$$L(t) = L_\infty(1 - \exp(-kt)) \tag{3}$$
$$\frac{dL}{dt} = L_\infty k \exp(-kt) \tag{4}$$

Take the derivative of $F(t)$ and set equal to zero to optimize the function:

$$F'(t) = -m \exp(-mt) L(t)^b + \exp(-mt) b L(t)^{b-1} \frac{dL}{dt}$$

Now set $F'(t) = 0$

$$F'(t) = 0 = \exp(-mt) L(t)^b (-m + bl(t)^{-1} \frac{dL}{dt})$$

Divide through by $\exp(-mt)L(t)^b$, continue to solve for t.

$$m = bL(t)^{-1} \frac{dL}{dt}$$
$$mL(t) = b \frac{dL}{dt}$$

Now substitute in our $L(t)$ and $\frac{dL}{dt}$ [Equations 3 and 4] to continue solving for t.

$$
\begin{aligned}
mL_\infty(1 - \exp(-kt)) &= bL_\infty k \exp(-kt) \\
\frac{m - m\exp(-kt)}{\exp(-kt)} &= \frac{bk\exp(-kt)}{\exp(-kt)} \\
m\exp(-kt) - m &= bk \\
\exp(-kt) &= \frac{bk + m}{m} \\
t^* &= \frac{1}{k} ln(\frac{bk + m}{m})
\end{aligned}
$$

Exercise 2.4 (E/M)
Show that size at maturity is given by $L(t_m) = L_\infty(\frac{bk}{m+bk}) = L_\infty(\frac{b}{b+\frac{m}{k}})$.

$$
\begin{aligned}
L(t) &= L_\infty(1 - \exp(kt)) \\
t_m &= \frac{1}{k}\log(\frac{m + bk}{m}) \\
L(t_m) &= L_\infty(1 - \exp(k(\frac{1}{k}log(\frac{m + bk}{m})))) \\
&= L_\infty[1 - \exp(-log(\frac{m + bk}{m}))] \\
&= L_\infty(1 - \frac{1}{\frac{m+bk}{m}}) \\
&= L_\infty(\frac{m + bk - m}{m + bk}) \\
&= L_\infty(\frac{bk}{m + bk}) \\
&= L_\infty(\frac{b}{b + \frac{m}{k}})
\end{aligned}
$$

Exercise 2.5 (E)

$$\frac{dW}{dt} = aW^{\frac{3}{4}} - bW$$

set $W = H^n$ then $\frac{dW}{dt} = nH^{n-1}\frac{dH}{dt}$.

$$\frac{dW}{dt} = aH^{\frac{3n}{4}} - bH^n$$

$$
\begin{aligned}
\frac{nH^{n-1}\frac{dH}{dt}}{nH^{n-1}} &= \frac{aH^{\frac{3n}{4}} - bH^n}{nH^{n-1}} \\
\frac{dH}{dt} &= (aH^{\frac{-n}{4}+1} - bH)\frac{1}{n}
\end{aligned}
$$

if $n = 4$ then

$$
\begin{aligned}
\frac{dH}{dt} &= \frac{1}{4}(a - bH) \\
\int \frac{1}{a - bH} dH &= \int \frac{1}{4} dt \\
-\ln a - bH &= \frac{1}{4}t + c \\
a - bH &= \exp(\frac{-1}{4}t + c) \\
-bH &= \exp(\frac{-1}{4}t + c) - a \\
H(t) &= \frac{\exp^{\frac{1}{4}t+c} - a}{b}
\end{aligned}
$$

Put in initial value and return back to W.

$$
H(0) = W(0)^{\frac{1}{4}}
$$

To put the equation back in the form similar to the Von Bertalanffy equation of length, we use $n = 4$ and write $\frac{dH}{dt} = \frac{1}{4}(a - bH)$.

Exercise 2.6 Temporal Variation (E/M)
Given

$$
\begin{aligned}
\lambda_1 &< 1 \\
N(t) &= \lambda_1^{pt}\lambda_2^{(1-p)t}N(0) \\
&= (\lambda_1^{p}\lambda_2^{1-p})^t N(0)
\end{aligned}
$$

For the population to increase we need $[\lambda_1^p\lambda_2^{(1-p)}]^t > 1$.
Assume

$$
\lambda_1^p\lambda_2^{(1-p)} > 1
$$

Then we can take the *log* and manipulate the inequality.

$$
\begin{aligned}
p\log(\lambda_1) + (1-p)\log(\lambda_2) &> 0 \\
(1-p)\log(\lambda_2) &> -p\log(\lambda_1) \\
\lambda_2^{1-p} &> \frac{1}{\lambda_1^p} \\
\lambda_2^{1-p} &> \lambda_1^{-p} \\
\lambda_2 &> \lambda_1^{\frac{-p}{(1-p)}}
\end{aligned}
$$

For a numerical example assume that the per capita growth rate for the poor habitat, λ_1 is set at $\frac{3}{4}$, and that the total fraction of poor habitat, p, is $\frac{1}{2}$. Thus the total fraction of good habitat, λ_2, would be $1-p=\frac{1}{2}$.So for the population to increase the per capita growth rate for the good habitat, λ_2, would need to be greater than $(\frac{3}{4})^{\frac{-.5}{1-.5}} = \frac{3}{4}^{-1} = \frac{4}{3}$.

Exercise 2.7 Logistic Equations (M)

$$
\begin{aligned}
\frac{dt(\frac{dN(t)}{dt})}{N(t)(1-\frac{N(t)}{K})} &= \frac{rN(t)(1-\frac{N(t)}{K})dt}{N(t)(1-\frac{N(t)}{K})} \\
\frac{dN}{N(1-\frac{N}{K})} &= rdt
\end{aligned}
$$

partial fraction decomposition:

$$
\begin{aligned}
\frac{1}{N(1-\frac{N}{K})} &= \frac{A}{N} + \frac{B}{1-\frac{N}{K}} \\
1 &= A(1-\frac{N}{K}) + BN \\
1 &= A - \frac{A}{K}N + BN \\
0 \cdot N + 1 &= A + N(B - \frac{1}{K}A)
\end{aligned}
$$

$\rightarrow A = 1$, $B - \frac{1}{K}A = 0$, so $B = \frac{1}{K}$
Integrate:

$$\int \frac{1}{N}dN + \int \frac{1}{K-N}dN = \int rdt$$

$$= \ln N - \ln(K - N) = rt + c$$
$$= \ln(\frac{N}{K - N}) = rt + c$$

Initial conditions: $t = 0 \Rightarrow N(t) = N(0)$

$$ln(\frac{N(0)}{K - N(0)}) = c$$

Plug constant back in, and take the exponential of each side.

$$\exp(ln(\frac{N(0)}{K - N(0)})) = \exp(rt + ln(\frac{N(0)}{K - N(0)}))$$
$$\frac{N(t)}{K - N(t)} = \exp(rt)\frac{N(0)}{K - N(0)}$$
$$N(t) = \frac{N(0)}{K - N(0)}\exp(rt)(K - N(t))$$
$$N(t) = \frac{N(0)K\exp(rt)}{K - N(0)} - \frac{N(0)\exp(rt)N(t)}{K - N(0)}$$

Now isolate and solve for $N(t)$.

$$\frac{N(0)\exp(rt)N(t)}{K - N(0)} + \frac{(K - N(0))N(t)}{KN(0)} = \frac{N(0)K\exp(rt)}{K - N(0)}$$
$$N(t)[\frac{K - N(0) + N(0)\exp(rt)}{K - N(0)} = \frac{N(0)K\exp(rt)}{K - N(0)}$$
$$N(t) = \frac{K - N(0)}{k - N(0) + N(0) + N(0)\exp(rt)} \cdot \frac{N(0)K\exp(rt)}{K - N(0)}$$

Now simplify the equation, and rearrange the denominator:

$$N(t) = \frac{N(0)K\exp(rt)}{K + N(0)(\exp(rt) - 1)}$$

Now we'll check this result. As $t \to \infty$ then $N(t) \to K$. This makes sense since as time goes to infinity the population size will level off at its carrying capacity, K. At $t = 0$, then $N(t)$ will be equal to $N(0)$ as it must be.

Exercise 2.8 Ricker Map (E/M)

$$\frac{dN}{dt} = lim_{dt \to 0} \frac{N(t+dt) - N(t)}{dt} = rN(t)(1 - \frac{N(t)}{K})$$

Now let us ignore the limiting process in the previous equation, and simply set $dt = 1$. If we do that then we can write

$$N(t+1) - N(t) = rN(t)(1 - \frac{N(t)}{K})$$

This equation is called the logistic map.

We can continue to switch $N(t+1)$ to a different form by using $N(t+1) = AN(t)f^{N(t)}$, and since we often set $f(n) = \exp(-bN)$ we obtain:

$$N(t+1) = N(t)\exp(r(1 - \frac{N(t)}{K})) = N(t)\exp(r)\exp(-\frac{rN}{K})$$
$$N(t+1) = AN(t)\exp(-bN)$$
$$A = \exp(r)$$
$$b = \frac{r}{K}$$

a) $f^{N(t)} = \exp(-bN)$, relationship between f and b is $f = \frac{1}{\exp(b)}$.

b)Ricker Map $N(t+1) = AN(t)\exp(-bN(t)) = AN(t)\frac{1}{\exp(bN(t))}$. $N(t)$ can grow as large as it wants since $\exp(bN(t)) > 1$, therefore $\frac{a}{\exp(bN(t))} > 1$ and $AN(t)\frac{1}{\exp(bN(t))}$ so it will never be less than zero unlike in the equation $N(t+1) = N(t) + rN(t)(1 - \frac{N(t)}{K})$ where if $N(t) > K$ then $N(t+1)$ could be less than $N(t)$ and even negative.

c)A Taylor Expansion can be used to show the connection between the Ricker and Logistic Map. The Taylor Expansion of $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + ...$
The Ricker Map:

$$N(t+1) = AN(t)\exp(-bN(t))$$
$$N(t+1) = AN(t)[1 - bN(t) + ...]$$

so that

$$N(t+1) \approx AN(t)(1-bN(t))$$

from equation 2.29 we know

$$N(t+1) = (r+1)N(t) - \frac{r}{k}N(t)^2$$

and the coefficients for the Logistic Map would be: $A = r+1$, $b = \frac{r}{K}$
.

Exercise 2.9 (E)
Two dimensional equations and classification of steady states
Equations from 2.37 in the book:

$$\frac{dx}{dt} = Ax + By \tag{5}$$
$$\frac{dy}{dt} = Cx + Dy \tag{6}$$

So we begin by taking the second derivative with respect to x, and obtain:

$$\frac{d^2x}{dt^2} = A\frac{dx}{dt} + B\frac{dy}{dt}$$

Next we substitute in $\frac{dy}{dt} = Cx + Dy$ and $y = (\frac{dx}{dt} - Ax)(\frac{1}{B})$ obtained from Equation 5.

$$\frac{d^2x}{dt^2} = A\frac{dx}{dt} + B(Cx + Dy)$$
$$\frac{d^2x}{dt^2} = A\frac{dx}{dt} + B(Cx + D\frac{1}{B})(\frac{dx}{dt} - Ax)$$

Now we consolidate the coefficients for x, $\frac{dx}{dt}$, and $\frac{d^2x}{dt^2}$.

$$\frac{d^2x}{dt^2} = A\frac{dx}{dt} + BC + BD\frac{1}{B}(\frac{dx}{dt} - Ax)$$

$$= -\frac{d^2x}{dt^2} + A\frac{dx}{dt} + BCx + D\frac{dx}{dt} - ADx$$
$$= -\frac{d^2x}{dt^2} + (A+D)\frac{dx}{dt} + (BC - DA)x$$
$$= \frac{d^2x}{dt^2} - (A+D)\frac{dx}{dt} + (DA - BC)x$$

Exercise 2.10 (E/M)
We know that

$$\frac{d^2x_1}{dt^2} - (A+D)\frac{dx_1}{dt} + (AD - BC)x_1 = 0 \tag{7}$$
$$\frac{d^2x_2}{dt^2} - (A+D)\frac{dx_2}{dt} + (AD - BC)x_2 = 0 \tag{8}$$
$$X(t) = ax_1(t) + bx_2(t) \tag{9}$$

Where a and b are constants.
We need to show that

$$\frac{d^2X}{dt^2} - (A+D)\frac{dX}{dt} + (AD - BC)X = 0 \tag{10}$$

So we begin by substituting Equation 9 into Equation 10.

$$\frac{d^2(ax_1 + bx_2)}{dt} - (A+D)\frac{d(ax_1 + bx_2)}{dt} + (AD - BC)(ax_1 + bx_2)$$

Continue by consolidating variables and factoring out a and b from the variables.

$$\frac{d^2ax_1}{dt^2} + \frac{d^2bx_2}{dt^2} - (A+D)(\frac{dax_1}{dt} + \frac{dbx_2}{dt}) + (AD - BC)(ax_1 + bx_2) = 0$$

$$0 = \frac{ad^2x_1}{dt} + \frac{bd^2x^2}{dt} - \frac{Aadx_1}{dt} - \frac{Abdx^2}{dt} - \frac{DAdx_1}{dt} - \frac{DBdx_2}{dt} + ADax_1 + ADbx_2 - BCax_1 - BCbx_2$$
$$0 = a[\frac{d^2x_1}{dt} - \frac{ADx_1}{dt} - \frac{Ddx_1}{dt} + ADx_1 - BCx_1] + b[\frac{d^2x_2}{dt} - \frac{ADx_2}{dt} - \frac{Ddx_2}{dt} + ADx_2 - BCx_2]$$

By Equations 7 and 8, both terms in brackets are equivalent 0. Therefore $X(t)$ is a solution.

Exercise 2.11 (M)
Show that if λ is a solution of $\lambda = (\frac{(A+D) \pm \sqrt{(A-D)^2 + 4BC}}{2})$ and that if we set $u = B$ and $v = \lambda - A$ that Equations 11 and 12 are satisfied.

$$(a-\lambda)u + Bv = 0 \tag{11}$$
$$Cu + (D-\lambda)v = 0 \tag{12}$$

Substitute in u, v.
In equation 11, $-vu + uv = 0$; the equation is satisfied.
For equation 12;

$$\begin{aligned}
CB + (D-\lambda)(\lambda - A) &= 0 \\
CB + D\lambda - DA - \lambda^2 + A\lambda &= 0 \\
-\lambda^2 + (A+D)\lambda - DA + CB &= 0 \\
\lambda^2 - (A+D)\lambda + (DA - CB) &= 0
\end{aligned}$$

This is in the form of the characteristic polynomial satisfying $x = \frac{b \pm \sqrt{b^2-4ac}}{2a}$. So $a = 1$, $b = A + D$, $c = DA - CB$.
Therefore

$$\begin{aligned}
\lambda = \frac{(A+D) \pm \sqrt{(A+D)^2 - 4(AD-BC)}}{2} \\
= \frac{(A+D) \pm \sqrt{(A+D)^2 - 4BC}}{2} \qquad (13)
\end{aligned}$$

Thus Equation 12 is satisfied.

Exercise 2.12 (M-H)

$$\begin{aligned}
\frac{dV}{dt} &= rV(1 - \frac{V}{K}) - bVP \\
\frac{dP}{dt} &= cVP - mP
\end{aligned}$$

Converting to per capita growth rates we have

$$\begin{aligned}
\frac{dV}{dt}\frac{1}{V} &= r(1 - \frac{V}{K}) - bP \\
\frac{dP}{dt}\frac{1}{P} &= cV - m
\end{aligned}$$

a)units of parameters
$r = \frac{1}{time}, \frac{1}{years}$
V=number of prey
K=number of prey, carrying capacity
P=number of predators
b=years^{-1}number of predators^{-1}
$c = (\frac{1}{time})(\frac{1}{prey})$
$m = \frac{1}{time}$
$\frac{dV}{dt} = \frac{prey}{time}$
$\frac{dP}{dt} = \frac{predators}{time}$

b)The differential equation describing prey's population levels includes what the population would be without the presence of predetors, contingent on the carrying capacity minus the encounter rate with predators and how many predators there are. Similarly with the predator quation, their population is contingent on the number of prey available and how often the predators encounter prey minus their populations natural mortality rate.

The c, b relationship:
(rate at which prey disappear)/(rate at which predators appear) $= \frac{bPV}{cPV} = \frac{b}{c}$. $\frac{b}{c}$ is the number of prey needed to 'make' a predator.

c) isocline analysis: solve for $\frac{dV}{dt} = \frac{dP}{dt} = 0$

$$\frac{dV}{dt} = V(r(1 - \frac{V}{K}) - bP)$$

If $\frac{dV}{dt} = 0$ either $V = 0$, or:

$$r(1 - \frac{V}{K}) - bP = 0$$
$$P = \frac{r}{b}(1 - \frac{V}{K})$$

Therefor the y-intercept is $\frac{r}{b}$, and the x-intercept is K.

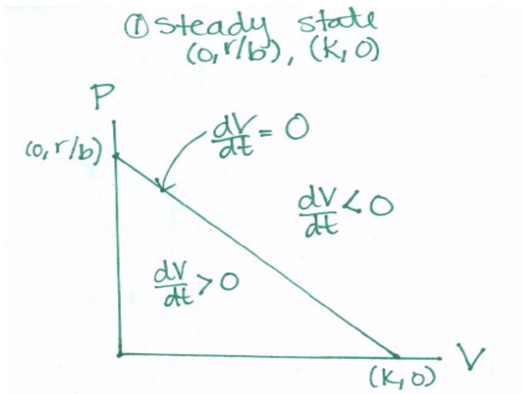


Figure 2. Isocline Analysis when $\frac{dV}{dt} = 0$

Second steady state:

$$\frac{dP}{dt} = cPV - mP = 0$$

If $\frac{dP}{dt} = 0$ either $P = 0$, or:

$$V = \frac{m}{c}$$

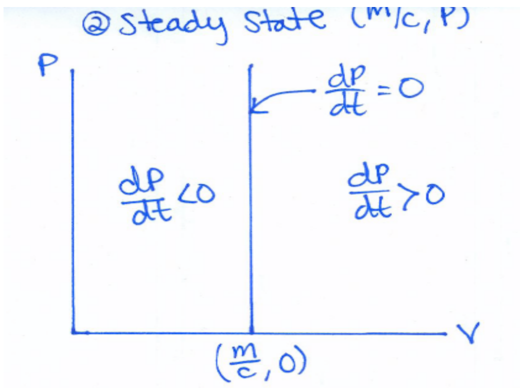


Figure 3. Isocline Analysis when $\frac{dP}{dt} = 0$. The steady state is $(\frac{m}{c}, P)$.

There are two cases for the isocline analysis depending on the relationship between K and $\frac{m}{c}$.

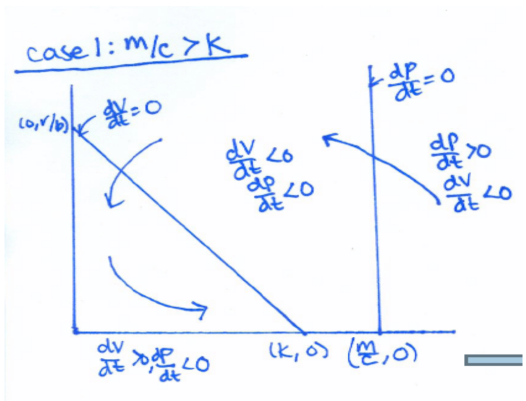


Figure 4. First steady state where $\frac{m}{c} > K$. This case leads to the extinction of the predators.

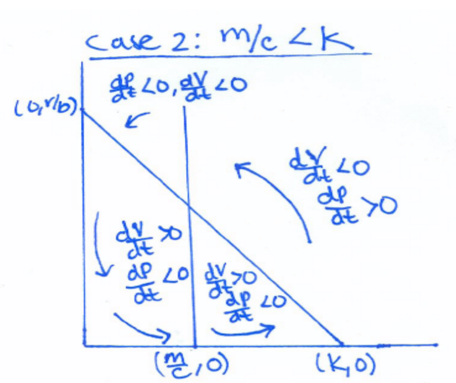


Figure 5. Second steady state where $\frac{m}{c} < K$. This scenerio leads to predator-prey cycles.

d)Classify the steady states

$$f(V,P) = rV(1-\frac{V}{K}) - bPV$$
$$g(V,P) = cPV - mP$$

The partial derivatives of the functions f and g are:

$$f_V = r - \frac{2rV}{K} - bP$$
$$f_P = -bV$$
$$g_V = cP$$
$$g_P = cV - m$$

The Jacobian is:

$$J(V,P) = \begin{bmatrix} r - \frac{2rV}{K} - bP & -bV \\ cP & cV - m \end{bmatrix}$$

$$J(V,P) = \begin{bmatrix} r - \frac{2rV}{K} - bP & -bV \\ cP & cV - m \end{bmatrix}$$

At the steady state $(\frac{m}{c}, \frac{r}{b}(1-\frac{\frac{m}{c}}{K}))$.

$$J(\frac{m}{c}, \frac{r}{b}(1-\frac{\frac{m}{c}}{K})) = \begin{bmatrix} r - \frac{2r\frac{m}{c}}{K} - b\frac{r}{b}(1-\frac{\frac{m}{c}}{K}) & -b\frac{m}{c} \\ c\frac{r}{b}(1-\frac{\frac{m}{c}}{K}) & c\frac{m}{c} - m \end{bmatrix}$$

$$J(\frac{m}{c}, \frac{r}{b}(1-\frac{\frac{m}{c}}{K})) = \begin{bmatrix} r - \frac{2r\frac{m}{c}}{K} - r(1-\frac{\frac{m}{c}}{K}) & -b\frac{m}{c} \\ c(\frac{r}{b}(1-\frac{\frac{m}{c}}{K}) & 0 \end{bmatrix}$$

First Analysis: Let K $\to \infty$, $K > \frac{m}{c}$ then $\frac{2r\frac{m}{c}}{K} \to 0$, and $\frac{\frac{m}{c}}{K} \to 0$ then

$$J(\frac{m}{c}, \frac{r}{b}) = \begin{bmatrix} r - r & -b\frac{m}{c} \\ \frac{cr}{b} & 0 \end{bmatrix}$$

$$J(\frac{m}{c}, \frac{r}{b}) = \begin{bmatrix} 0 & -b\frac{m}{c} \\ \frac{cr}{b} & 0 \end{bmatrix}$$

Eigenvalues are roots of the characteristic equation.

$\lambda_{1,2} = \frac{(A+D) \pm \sqrt{(A+D)^2 + 4BC}}{2}$

$$A = 0$$
$$B = \frac{-bm}{c}$$
$$C = \frac{cr}{b}$$
$$D = 0$$
$$\lambda_{1,2} = \frac{+}{-}\sqrt{-4\frac{bm}{c} \cdot \frac{cr}{b}} = \frac{+}{-}\sqrt{-4mr}$$

Values under the radical tell us about the oscillations about the steady state. $\lambda_{1,2}$ is purely complex if K $\to \infty$. λ becomes purely imaginary and pertubations from the steady states lead to oscillations of the population values.

Second Analysis:

$$J(\frac{m}{c}, \frac{r}{b}(1 - \frac{\frac{m}{c}}{K})) = \begin{bmatrix} r - \frac{2r\frac{m}{c}}{K} - b\frac{r}{b}(1 - \frac{\frac{m}{c}}{K}) & -b\frac{m}{c} \\ c\frac{r}{b}(1 - \frac{\frac{m}{c}}{K}) & c\frac{m}{c} - m \end{bmatrix}$$

$$J(\frac{m}{c}, \frac{r}{b}(1 - \frac{\frac{m}{c}}{K})) = \begin{bmatrix} -\frac{r\frac{m}{c}}{K} & \frac{bmK}{cK} \\ \frac{cr}{b}(1 - \frac{\frac{m}{c}}{K}) & 0 \end{bmatrix}$$

input into standard equation:

$$\lambda = \frac{-rm}{cK} \frac{+}{-} \sqrt{(\frac{-rm}{cK})^2 + 4(\frac{-bmK}{cK})(\frac{cr}{b})(1 - \frac{\frac{m}{c}}{K})}$$

The $(\frac{-rm}{cK})^2$ shows that the steady state will be stable.

If $\frac{m}{c} > K \rightarrow (1 - \frac{\frac{m}{c}}{K}) < 0$
If $\frac{m}{c} < K \rightarrow (1 - \frac{\frac{m}{c}}{K}) > 0$
We need $K > \frac{m}{c}$ for the predators to persist. Thus $1 - \frac{\frac{m}{c}}{K} > 0$ always.
The eigenvalues are real if

$$\begin{aligned} (\frac{rm}{ck})^2 > 4(\frac{-bmK}{cK})(\frac{cr}{b})(1 - \frac{\frac{m}{c}}{K}) \\ \frac{rm}{(ck)^2} > 2\sqrt{1 - \frac{\frac{m}{c}}{K}} \\ rm > 2(ck)^2\sqrt{1 - \frac{\frac{m}{c}}{K}} \\ r > 2(ck)^2\sqrt{1 - \frac{\frac{m}{c}}{K}} \end{aligned}$$

Otherwise, the eigenvalues are complex.

What do the eigenvalues tell us that the isocline analysis doesn't?

We can see the oscillation but not the stable steady state.

e) What happens to the eigen values as K goes to ∞?
Discussed above before the Second Analysis.

Exercise 2.13 (M)
Show that

$$p(x,t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp(\frac{-(x - vt)^2}{2\sigma^2 t}) \quad (14)$$

is a solution to

$$p_t = -vp_x + \frac{\sigma^2}{2} p_{xx} \tag{15}$$

Part 1: To begin let's solve p_t, $\frac{dp(x,t)}{dt}$, from Equation 14. Let $p(x,t) = c(t)\exp(u(x,t))$ where $c(t) = \frac{1}{\sqrt{2\sigma^2 t}}$ and $u(x,t) = \frac{-(x-vt)^2}{2\sigma^2 t}$.
Then

$$p_t = \frac{dc}{dt}\exp(u) + c(t)\frac{du}{dt}\exp(u)$$

Now we'll solve $\frac{dc}{dt}$, and $\frac{du}{dt}$ so we can plug them into our p_t equation and then compare it to Equation 15.

$$\frac{dc}{dt} = -\frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}} \cdot t^{\frac{-3}{2}}$$
$$\frac{du}{dt} = \frac{2(x-vt)v(2\sigma^2 t) + 2\sigma^2(x-vt)^2}{(2\sigma^2 t)^2}$$

Therefore
$$p_t = -\frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}} \cdot t^{\frac{-3}{2}} \cdot \exp(\frac{-(x-vt)^2}{2\sigma^2 t})^2) + \frac{1}{2\sigma^2 t} \cdot \frac{2(x-vt)v(2\sigma^2 t) + 2\sigma^2(x-vt)^2}{(2\sigma^2 t)^2} \cdot \exp(\frac{-(x-vt)^2}{2\sigma^2 t})$$
Simplifying we obtain:
$$p_t = \frac{-v(-2(x-vt)\exp(u)(2\sigma^2 t)) - \sigma^2\exp(u)(2\sigma^2 t) + 2\sigma^2(x-vt)^2\exp(u)}{(2\pi\sigma^2 t)^{\frac{1}{2}}(2\sigma^2 t)^2}$$
Part 2: Now we evaluate p_t from Equation 15 and compare it to our last answer.
Since $p_t = -vp_x + \frac{\sigma^2}{2}p_{xx}$, we will solve for p_x and p_{xx} and plug them back into our equation.

$$p_x = c(t)u_x\exp(u)$$
$$p_{xx} = c(t)[u_{xx}\exp(u) + u_x^2\exp(u)]$$

Now we solve for u_x, u_{xx}, and u_x^2 to complete our p_x and p_{xx} equations.

$$u_x = \frac{-2(x-vt)}{2\sigma^2 t}$$
$$u_{xx} = \frac{-2}{2\sigma^2 t}$$
$$(u_x)^2 = (\frac{-2(x-vt)}{2\sigma^2 t})^2$$

Now plug those back into the p_x and p_{xx} equations and obtain:

$$p_x = \frac{1}{\sqrt{2\sigma^2 t}} \cdot \frac{-2(x-vt)}{2\sigma^2 t} \cdot \exp(\frac{-(x-vt)^2}{2\sigma^2 t})$$
$$p_{xx} = \frac{1}{\sqrt{2\sigma^2 t}}[\frac{-2}{2\sigma^2 t} \cdot \exp(\frac{-(x-vt)^2}{2\sigma^2 t}) + (\frac{-2(x-vt)}{2\sigma^2 t})^2 \cdot \exp(\frac{-(x-vt)^2}{2\sigma^2 t})]$$

Now putting the previous equations into Equation 15 we obtain:

$$p_t = -v\frac{1}{\sqrt{2\sigma^2 t}} \cdot \frac{-2(x-vt)}{2\sigma^2 t} \cdot \exp(\frac{-(x-vt)^2}{2\sigma^2 t}) +$$

$$\frac{\sigma^2}{2}\frac{1}{\sqrt{2\sigma^2 t}}[\frac{-2}{2\sigma^2 t} \cdot \exp(\frac{-(x-vt)^2}{2\sigma^2 t}) + (\frac{-2(x-vt)}{2\sigma^2 t})^2 \cdot \exp(\frac{-(x-vt)^2}{2\sigma^2 t})]$$

Simplifying we obtain:

$$p_t = \frac{-v(-2(x-vt)\exp(u)(2\sigma^2 t)) - \sigma^2 \exp(u)(2\sigma^2 t) + 2\sigma^2 (x-vt)^2 \exp(u)}{(2\pi\sigma^2 t)^{\frac{1}{2}}(2\sigma^2 t)^2} \tag{16}$$

This is the same as out p_t obtained from Part 1. Therefore $p(x,t) = \frac{1}{\sqrt{2\pi\sigma^2 t}}\exp(\frac{-(x-vt)^2}{2\sigma^2 t})$ is a solution to $p_t = -vp_x + \frac{\sigma^2}{2}p_{xx}$.

Exercise 3.1 (E)

What are the associated z_k and p_k when the fair die is thrown twice and the results summed?

For the discrete random variable Z that can take a set of values $\{z_k\}$ we introduce probabilitilties p_k defined by $Pr\{Z = z_k\} = p_k$. So p_k =the number of ways to get outcome k over the total number of outcomes, 36.

Sample Calculation:
For $z_6 = 7$, Since $7 = 1{+}6, 2{+}5, 3{+}4, 4{+}3, 5{+}2, 6{+}1$ there are six possible ways to obtain z_6.

$z_1 = 2$, $p = \frac{1}{36}$
$z_2 = 3$, $p_2 = \frac{2}{36}$
$z_3 = 4$, $p_3 = \frac{3}{36}$
$z_4 = 5$, $p_4 = \frac{4}{36}$
$z_5 = 6$, $p_5 = \frac{5}{36}$

$z_6 = 7$, $p_6 = \frac{6}{36}$
$z_7 = 8$, $p_7 = \frac{5}{36}$
$z_8 = 9$, $p_8 = \frac{4}{36}$
$z_9 = 10$, $p_9 = \frac{3}{36}$
$z_{10} = 11$, $p_{10} = \frac{2}{36}$
$z_{11} = 12$, $p_{11} = \frac{1}{36}$

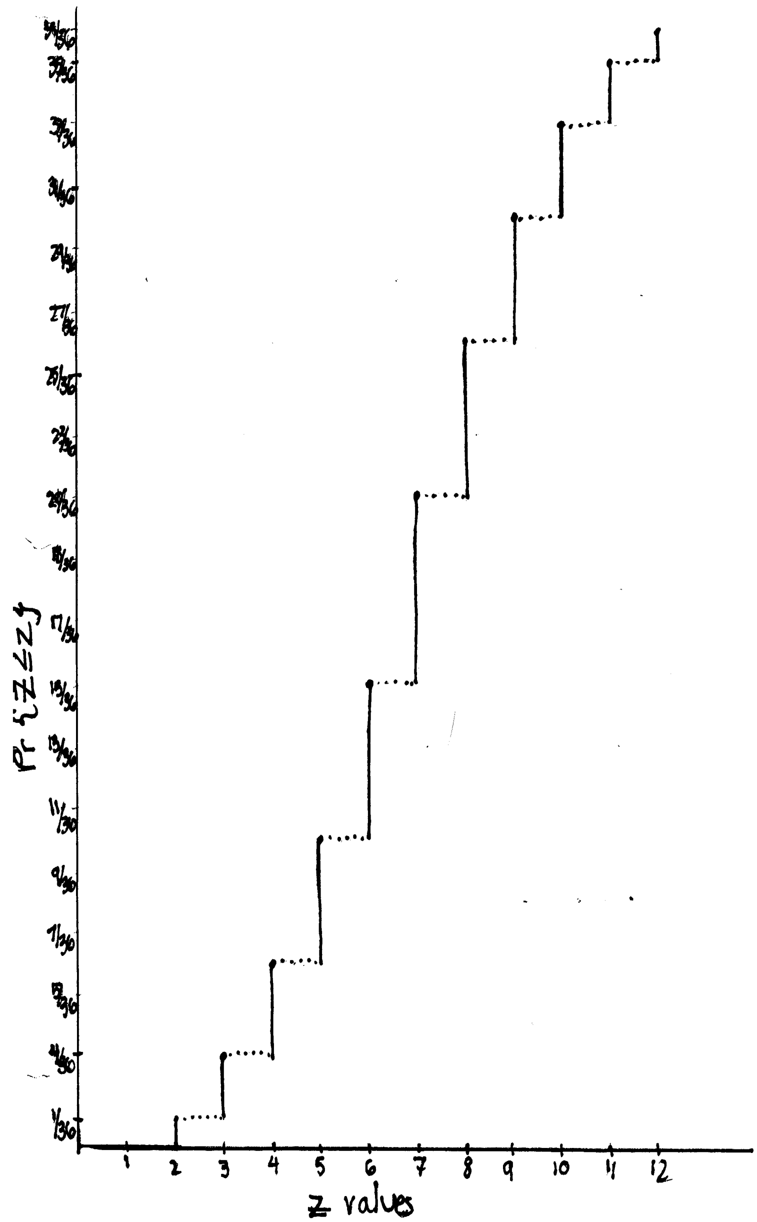


Figure 6. Probability Distribution Function of Die Thrown Twice

Exercise 3.2 (E)

$Pr\{Z \leq z\}$, Z is the sum of two die.

$Pr\{Z \leq z\} = \sum_{k=1}^{K(z)} p_k$, where $K(z)$=the value of k so that the sum is less that or equal to z.

Example Calculation

$Pr\{Z \leq 3\} = \sum_{k=1}^{K(3)} p_k = p_1 + p_2 = \frac{1}{36} + \frac{2}{36} = \frac{3}{36}$

$$Pr\{Z < 2\} = 0$$
$$Pr\{Z \leq 2\} = \sum_{k=1}^{K(2)} = \frac{1}{36}$$
$$Pr\{Z \leq 3\} = \sum_{k=1}^{K(3)} = \frac{3}{36}$$
$$Pr\{Z \leq 4\} = \sum_{k=1}^{K(4)} = \frac{6}{36}$$
$$Pr\{Z \leq 5\} = \sum_{k=1}^{K(5)} = \frac{10}{36}$$
$$Pr\{Z \leq 6\} = \sum_{k=1}^{K(6)} = \frac{15}{36}$$
$$Pr\{Z \leq 7\} = \sum_{k=1}^{K(7)} = \frac{21}{36}$$
$$Pr\{Z \leq 8\} = \sum_{k=1}^{K(8)} = \frac{26}{36}$$
$$Pr\{Z \leq 9\} = \sum_{k=1}^{K(9)} = \frac{30}{36}$$
$$Pr\{Z \leq 10\} = \sum_{k=1}^{K(10)} = \frac{33}{36}$$
$$Pr\{Z \leq 11\} = \sum_{k=1}^{K(11)} = \frac{35}{36}$$
$$Pr\{Z \leq 12\} = \sum_{k=1}^{K(12)} = \frac{36}{36}$$
$$Pr\{Z > 12\} = 0$$

Exercise 3.3 (E)

For a continuous random variable, the variance is $Var\{Z\} = \int (z - E\{Z\})^2 f(z)dz$. Show that an equivalent definition of variance is $Var\{Z\} = E\{Z^2\} - E^2\{Z\}$ where we define $E\{Z^2\} = \int z^2 f(z)dz$.

$$\begin{aligned} Var\{Z\} &= \int (z - E\{Z\})^2 f(z)dz \\ &= \int (z^2 - 2zE\{Z\} + E^2\{Z\})f(z)dz \\ &= \int z^2 f(z)dz - 2E\{Z\}\int zf(z)dz + E^2\{Z\} \\ &= E\{Z^2\} - 2E\{Z\}E\{Z\} - E^2\{Z\} \\ &= E\{Z^2\} - 2E^2\{Z\} + E^2\{Z\} \\ &= E\{Z^2\} - E^2\{Z\} \end{aligned}$$

Exercise 3.4 (E)

Friend's inspection Exercise:
One of my friends, Tracy, is another math major. When asked to identify the most variable series she chose series C. For the least variable she guessed series B.

The other person I asked was the librarian at the science library, named Barb. She was less than thrilled to examine the numbers, but when she did she guessed the following:
Most variable series- series B.
Least variable- series A.

Now for the actual calculations!

<u>Series A</u>
$45, 32, 12, 23, 26, 27, 39$
Mean$= \frac{45+32+12+23+26+27+39}{7} = 29.14$
Variance$= \frac{(45-29)^2+(32-29)^2+(12-29)^2+(23-29)^2+(26-29)^2+(27-29)^2+(39-29)^2}{7} = 100$
Standard Deviation$= \sqrt{Var(Z)} = \sqrt{100} = 10$
Coefficient of variation$= \frac{10}{29} = .34$

<u>Series B</u>
$1401, 1388, 1368, 1379, 1382, 1383, 1395$
Mean$= \frac{1401+1388+1368+1379+1382+1383+1395}{7} = 1,385$

Variance= $\frac{(1401-1385)^2+(1388-1385)^2+(1368-1385)^2+(1379-1385)^2+(1382-1385)^2+(1383-1385)^2+(1395-1385)^2}{7} = 100$
Standard Deviation= 10
Coefficient of Variation= $\frac{10}{1385} = .007$

<u>Series C</u>
$225, 160, 50, 115, 130, 135, 195$
Mean= $\frac{225+160+50+115+130+135+195}{7} = 144$
Variance= $\frac{(225-144)^2+(160-144)^2+(50-144)^2+(115-144)^2+(130-144)^2+(135-144)^2+(195-144)^2}{7} = 2,767$
Standard Deviation= $\sqrt{2767} = 52$
Coefficient of Variation= $\frac{52}{144} = .36$

By inspection it is not at all clear that the series A and B would have an identical standard deviation and variance since the numbers in series B are so much greater than in A. The standard deviation looks small relative to B. It is also interesting to note that the coefficient of variations is similar in series A and C. So their dispersion of a probability distribution is nearly equal.

Exercise 3.5 (M)
Show that

$$Var\{K\} = Np(1-p)$$

Since

$$Var\{K\} = E\{K^2\} - (E\{K\})^2 \tag{17}$$

and we already know that

$$E\{K\}^2 = Np \tag{18}$$

Then we'll continue by solving for $E\{K^2\}$.

$$E\{K^2\} = \sum_{k=0}^{N} k^2 \begin{pmatrix} N \\ K \end{pmatrix} p^k(1-p)^{N-k}$$

To process one factor of k with $j = k - 1$, use the same method used to obtain the mean.

$$\begin{aligned} E\{K^2\} &= Np \sum_{j=0}^{N-1} k \begin{pmatrix} N \\ K \end{pmatrix} p^j (1-p)^{N-k} \\ &= Np \sum_{j=0}^{N-1} (j+1) p^j (1-p)^{N-1-j} \end{aligned}$$

split the sum:

$$= Np(\sum_{j=0}^{N-1} j \begin{pmatrix} N-1 \\ j \end{pmatrix} p^j (1-p)^{N-1-j} + \sum_{j=0}^{N-1} \begin{pmatrix} N-1 \\ j \end{pmatrix} p^j (1-p)^{N-1-j})$$

then let $m = N - 1$

$$= Np(\sum_{j=0}^{m} j \begin{pmatrix} m \\ j \end{pmatrix} p^j (1-p)^{m-j} + \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} p^j (1-p)^{m-j})$$

the first sum is equal to mp, the second sum is equal to 1.

$$E\{K^2\} = Np(mp+1)$$

Now substitute back in $N - 1$ for m.

$$\begin{aligned} &= Np((N-1)p+1) \\ &= Np(Np - p + 1) \\ &= (Np)^2 - Np^2 + Np \end{aligned} \tag{19}$$

Going back to our initial equation, Equation 17, we plug in Equation 18 and 19 to obtain:

$$\begin{aligned} Var\{K\} &= (Np)^2 - Np^2 + Np - (Np)^2 \\ &= Np(p-1) \end{aligned}$$

Exercise 3.6 (E)
Show that $\frac{Pr\{K=k+1\}}{Pr\{K=k\}} = \frac{(N-k)p}{(k+1)(1-p)}$
.

From 3.22 we know

$$
\begin{aligned}
Pr\{K = k\} &= \frac{N!}{k!(N-k)!}p^k(1-p)^{N-k} \\
&= \frac{N!p^k(1-p)^N}{k!(N-k)!(1-p)^k} \\
&= \frac{N!p^k(1-p)^N}{k!(N-k)(N-k-1)!(1-p)^k}
\end{aligned}
$$

and we also know that

$$
\begin{aligned}
Pr\{K = K+1\} &= \frac{N!p^{k+1}(1-p)^N}{(k+1)!(N-(k+1))!(1-p)^{k+1}} \\
&= \frac{N!p^k p(1-p)^N}{(k+1)!k(N-(k+1))!(1-p)^k(1-p)}
\end{aligned}
$$

so that

$$
\frac{Pr\{K = k+1\}}{Pr\{K = k\}} = \frac{p(N-k)}{(k+1)(1-p)}
$$

Exercise 3.7 (E)
Show that the MLE for p is $\hat{p} = \frac{k}{N}$.

$$
\begin{aligned}
L(p|k, N) &= log(\frac{N}{k}) + klog(p) + (N-k)log(1-p) \\
\frac{dL}{dp} &= k(\frac{1}{p}) + (N-k)(\frac{1}{1-p})(-1)
\end{aligned}
$$

Set $\frac{dL}{dp} = 0$ and solve for $\hat{p}$.

$$
\begin{aligned}
0 &= \frac{k}{p} - \frac{N-k}{1-p} \\
0 &= \frac{k - pk - pN + pk}{1-p} \\
0 &= k - pk - pN + pk \\
\frac{-k}{-N} &= p \\
\hat{p} &= \frac{k}{N}
\end{aligned}
$$

Exercise 3.8 (E)
Show that setting $\frac{\hat{L}(N+1|k,p)}{\hat{L}(N|k,p)} = 1$ leads to the equation $\frac{(N+1)(1-p)}{N+1-k} = 1$. Solve this equation for N to obtain $\hat{N} = \frac{k}{p} - 1$. Does this accord with your intuition?

We know that

$$
\begin{aligned}
\hat{L}(N+1|k,p) &= \frac{(N+1)!p^k(1-p)^{(N+1)-k}}{k!(N-1-k)!} \\
&= \frac{(N+1)N!p^k(1-p)^N(1-p)}{k!(N+1-k)(N-k)!(1-p)^k} \\
= \hat{L}(N|k,p) &= \frac{N!p^k(1-p)^{N-k}}{k!(N-k)!} \\
&= \frac{N!p^k(1-p)^N}{k!(N-k)!(1-p)^k}
\end{aligned}
$$

So that

$$
\begin{aligned}
\frac{\hat{L}(N+1|k,p)}{\hat{L}(N|k,p)} = 1 &= \frac{(N+1)(1-p)}{N+1-k} \\
(N+1)(1-p) &= N+1-k \\
-1(N - Np + 1 - p) &= N+1-k \\
Np + p &= k \\
p(N+1) &= k
\end{aligned}
$$

$$N + 1 = \frac{k}{p}$$
$$\hat{N} = \frac{k}{p} - 1$$

Does this accord with your intuition?
Yes. For example if someone is flipping a coin and gets 10 heads then it's likely that they threw the dice 20 times.

Exercise 3.9 (M)
The probabity of having k events in t to $t + dt$ is the sum of having either $k - 1$ events in time 0 to t and 1 event in time t to $t + dt$ or k events in time 0 to t and none in time t to $t + dt$. Since they are mutually exlusive events we can write them as

$$P_k(t + dt) = P_{k-1}\lambda dt + P_k(t)(1 - \lambda dt) + o(dt)$$

We obtain the general equation by solving for $\frac{dP_k}{dt}$.

$$\begin{aligned} P_k(t + dt) - P_k(t) &= P_{k-1}(t)dt + o(dt) - \lambda P_k(t)dt \\ \frac{P_k(t + dt) - P_k(t)}{dt} &= \lambda P_{k-1}(t) - \lambda P_k(t) + \frac{o(dt)}{dt} \end{aligned}$$

let $dt \to 0$

$$\frac{dP_k}{dt} = \lambda P_{k-1}(t) - \lambda P_k(t) \tag{20}$$

Show that the solution of equation is $P_k(t) = \frac{(\lambda t)^k}{k!} \exp(-\lambda t)$

Start by evaluating $\lambda P_{k-1}(t)$

$$\begin{aligned} \lambda P_{k-1}(t) = \lambda \frac{(\lambda k)^{k-1} \exp(-\lambda t)}{(k-1)!} &= \frac{\lambda^k t^{k-1} \exp(-\lambda t)}{(k-1)!} \\ \lambda P_k(t) = \lambda \frac{(\lambda k)^k \exp(-\lambda t)}{k!} = \frac{\lambda^{k+1} t^k \exp(-\lambda t)}{k!} &= \frac{\lambda^{k+1} t^k \exp(-\lambda t)}{k(k-1)!} \end{aligned}$$

Thus

$$\lambda P_{k-1}(t) + \lambda P_k(t) = \frac{k\lambda^k t^{k-1}\exp(-\lambda t)}{k(k-1)!} - \frac{\lambda^k \lambda t^k \exp(-\lambda t)}{k(k+1)!}$$
$$= \frac{\lambda^k t^k \exp(-\lambda t)(kt^{-1} - \lambda)}{k!}$$

Now solve for $\frac{dP_k}{dt}$:

$$\frac{dP_k}{dt} = \frac{\lambda^k}{k!}\frac{d}{dt}t^k \exp(-\lambda t)$$
$$= \frac{\lambda^k}{k!}(kt^{k-1}\exp(-\lambda t) + t^k(-\lambda)\exp(-\lambda t))$$
$$= \frac{\lambda^k}{k!}(kt^{k-1}\exp(-\lambda t) - t^k\lambda\exp(-\lambda t))$$
$$= \frac{\lambda^k k t^k t^{-1}\exp(-\lambda t) - t^k\lambda^k\lambda\exp(-\lambda t)}{k!}$$
$$= \frac{\lambda^k t^k \exp(-\lambda t)(kt^{-1} - \lambda)}{k!}$$

Thus $P_k(t) = \frac{(\lambda t)^k}{k!}\exp(-\lambda t)$ is the solution to Equation 20.

Exercise 3.10 (E)
Show that $\frac{p_k(t)}{p_{k-1}(t)} > 1$ implies $\lambda t > k$.
By equation 3.40 we know $p_k(t) = \frac{\lambda t}{k}p_{k-1}(t)$. So the ratio $\frac{p_k(t)}{p_{k-1}(t)} = \frac{1}{\frac{\lambda t}{k}}$.
So since $\frac{1}{\frac{\lambda t}{k}} > 1$ then $1 > \frac{k}{\lambda t}$, thus $\lambda t > k$.

Exercise 3.11 (E)
$L(\lambda|k,t) = -\lambda t + klog(\lambda t) - log(k!) = -\lambda t + klog(\lambda) + klog(t) - log(k!)$
Show the MLE is $\hat{\lambda} = \frac{k}{t}$

$$\frac{dL}{d\lambda} = -t + \frac{k}{\lambda}$$

Now set $\frac{dL}{d\lambda} = 0$ and solve for $\hat{\lambda}$.

$$
\begin{aligned}
-t + \frac{k}{\lambda} &= 0 \\
\frac{k}{\lambda} &= t \\
k &= \frac{t}{\lambda} \\
\hat{\lambda} &= \frac{k}{t}
\end{aligned}
$$

Exercise 3.12 (E/M)
Show that $E\{\Lambda^2\} = \frac{\nu(\nu+1)}{\alpha^2}$

$$
\begin{aligned}
E\{\Lambda^2\} &= \int_0^\alpha \lambda^2 \frac{\alpha\nu}{\Gamma(\nu)} \exp(-\alpha\lambda)\lambda^{\nu-1} dt \\
&= \frac{\alpha\nu}{\Gamma(\nu)} \int \exp(-\alpha\lambda)\lambda^{\nu+1} dt \\
= \frac{\lambda^\nu}{\Gamma(\nu)} \frac{\Gamma(\nu+2)}{\alpha^{\nu+2}} &= \frac{\alpha^\nu \Gamma(\nu+2)}{\alpha^\nu \alpha^2 \Gamma(\nu)}
\end{aligned}
$$

Since

$$
\Gamma(\nu+1) = (\nu+1)\nu\Gamma(\nu)
$$

Then $E\{\Lambda^2\}$ will be

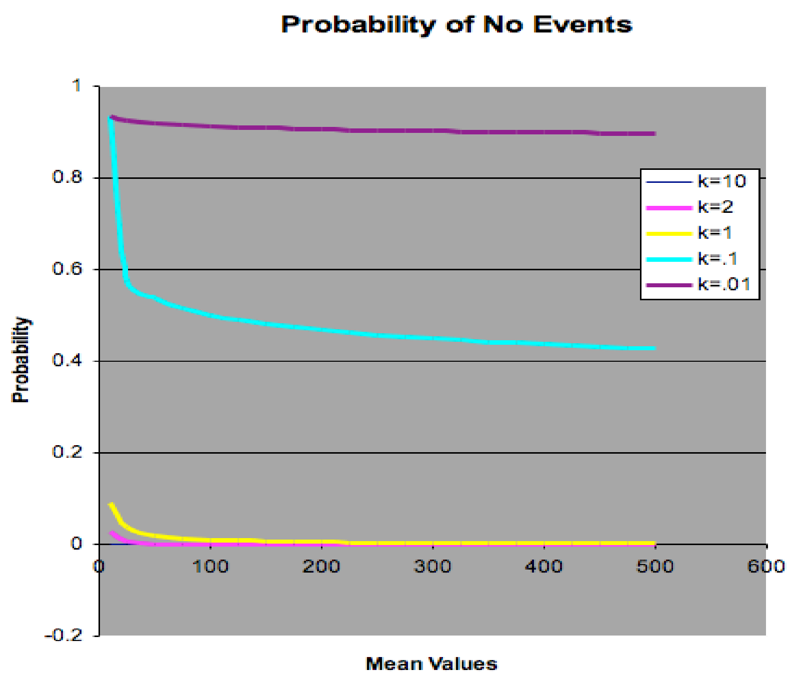
$$
= \frac{\nu(\nu+1)}{\alpha^2} = \frac{\nu^2+\nu}{\alpha^2}
$$

From exercise 3.3 we know $Var\{\Lambda\} = E\{\Lambda^2\} - (E\{\Lambda\})^2$. Also from Equation 3.56 we know that $E\{\Lambda\} = \frac{v}{\alpha}$. Thus

$$
\begin{aligned}
Var\{\Lambda\} = \frac{\nu^2+\nu}{\alpha^2} - (\frac{\nu}{\alpha})^2 = \frac{\nu^2+\nu-\nu^2}{\alpha^2} &= \frac{\nu}{\alpha^2} \\
\frac{\nu}{\alpha^2} = \frac{1}{\nu}\frac{\nu^2}{\alpha^2} &= \frac{1}{\nu}(E\{\Lambda\})^2
\end{aligned}
$$

Exercise 3.13 (E)

Interpretation of results of the graph 'Probability of No Events', with $k = .01, .1, 1, 2, 10$: The negative binomial distribution becomes more and more skewed as the dispersion parameter decreases. For $k = .1$ there is more than a 60% chance of zero events, even though the mean is about 10. Wow!



Exercise 3.14 (E)

X is normally distributed with mean 0 and variance 1. $f(x) = \frac{1}{\sqrt{2\pi}} \exp(\frac{-x^2}{2})$

Transformation $X = \frac{Y-\mu}{\sigma}$. For transformation the Jacobian is $\frac{dx}{dy}\frac{Y-\mu}{\sigma} = \frac{1}{\sigma}$.

$$f(X) = (\frac{1}{\sqrt{2\pi}}\exp(-(\frac{Y-\mu}{\sigma})^2))\frac{1}{\sigma} = \frac{1}{\sqrt{2\pi}}\sigma\exp(\frac{(-Y-u)^2}{2\sigma^2})$$

Exercise 3.15 (E)
Show that SSQ(m) is minimized when m is the sample $\frac{1}{n}\sum_{i=1}^{n} Y_i$.

$SSQ(m) = \sum_{i=1}^{n}(Y_i - m)^2$

To minimize $SSQ(m)$ we'll take the derivative, set it equal to 0, and solve for m. If we write out the sum we get:

$$\sum_{i=1}^{n}(Y_i - m)^2 = (y_1 - m)^2 + (y_2 - m)^2 + (y_3 - m)^2 + .. + (y_n - m)^2$$

Take the derivative with respect to m:

$$\frac{d}{dm}[(y_1 - m)^2 + (y_2 - m)^2 + (y_3 - m)^2 + ... + (y_n - m)^2] = 0$$
$$-2(y_1 - m) - 2(y_2 - m) - 2(y_3 - m) + ... - 2(y_n - m) = 0$$

Now go back to the summation form and solve for m.

$$2nm - 2(\sum y_i) = 0$$
$$nm - \sum y_i = 0$$
$$m = \frac{1}{n}\sum y_i$$

Exercise 3.16 (E/M)

$SSQ(a, b) = \sum_{i=1}^{N}(Y_i - a - bX_i)^2$

The derivative with respect to a:

$$\frac{d}{da}[\sum_{i=1}^{N}(Y_i - a - bX_i)^2] =$$

$$= \frac{d}{da}[(Y_1 - a - bX_1)^2 + (Y_2 - a - bX_2)^2 + (Y_3 - a - bX_3)^2 + ... + (Y_n - a - bX_n)^2]$$

$$= -2(Y_1 - a - bX_1) - 2(Y_2 - a - bX_2) - 2(Y_3 - a - bX_3) + ... - 2(Y_n - a - bX_n)$$

$$= 2na - 2(\sum_{i=1}^{N} Y_i) + 2b(\sum_{i=1}^{N} X_i)$$

Now we set $\frac{d}{da}SSQ = 0$ for a and obtain:

$$a = \frac{2(\sum_{i=1}^{N} Y_i) - 2b(\sum_{i=1}^{N} X_i)}{2n}$$

$$\hat{a} = \frac{1}{n}(\sum_{i=1}^{N} Y_i - b\sum_{i=1}^{N} X_i)$$

Now for the derivative with respect to b:

$$\frac{d}{db}SSQ[\sum_{i=1}^{N}(Y_i - a - bX_i)^2] =$$

$$\frac{d}{db}[(Y_1 - a - bX_1)^2 + (Y_2 - a - bX_2)^2 + (Y_3 - a - bX_3)^2 + ... + (Y_n - a - bX_n)^2]$$

$$\frac{d}{db}SSQ = -2X_1(Y_1 - a - bX_1) - 2X_2(Y_2 - a - bX_2) - 2X_3(Y_3 - a - bX_3) + ... - 2X_n(Y_n - a - bX_n)$$

$$\frac{d}{db}SSQ = -2n\sum_{i=1}^{N} X_iY_i + 2na\sum_{i=1}^{n} X_i - 2nb\sum_{i=1}^{n} X_i^2$$

Now we set $\frac{d}{db}SSQ = 0$ for b and obtain:

$$2nb\sum_{i=1}^{N} X_i^2 = 2n\sum_{i=1}^{N} X_iY_i - 2na\sum_{i=1}^{N} X_i$$

$$b = \frac{2n\sum_{i=1}^{N} X_iY_i - 2na\sum_{i=1}^{N} X_i}{2n\sum_{i=1}^{N} X_i^2}$$

$$\hat{b} = \frac{\sum_{i=1}^{N} X_iY_i - a\sum_{i=1}^{N} X_i}{\sum_{i=1}^{N} X_i^2}$$

Exercise 3.17 (E)

Show that if Y is defined by $Y = A\exp(\sigma X - \frac{1}{2}\sigma^2)$, then $E\{Y\} = A$ and $Var\{A\} = A^2(\exp(\sigma^2) - 1)$. Use equation 3.9 to find the expected value:

$$
\begin{aligned}
Z &= A\exp(\sigma X) \\
E\{Z\} &= A\exp(\frac{\sigma^2}{2}) \\
E\{Z^2\} &= A^2 E\{\exp(2\sigma X)\} \\
&= A^2\exp(2\sigma^2) \\
Y &= Z\exp(-\frac{1}{2}\sigma^2) \\
E\{Y\} &= E\{Z\exp(-\frac{1}{2}\sigma^2)\} \\
&= \exp(-\frac{1}{2}\sigma^2)E\{Z\}
\end{aligned} \tag{21}
$$

By equation 21 then

$$\exp(-\frac{1}{2}\sigma^2)E\{Z\} = A,$$

So

$$E\{Z\} = A\exp(\frac{1}{2}\sigma^2)$$

and

$$E\{Y\} = A$$

To find the variance we recall that $Var\{Y\} = E\{Y^2\} - E\{Y\}^2$.

$$
\begin{aligned}
E\{Y^2\} &= E\{Z^2\exp(-\sigma^2)\} \\
&= \exp(-\sigma^2)E\{Z^2\} = A^2\exp(\sigma^2)
\end{aligned}
$$

So then variance will be

$$
\begin{aligned}
Var\{Y\} &= A^2 \exp(\sigma^2) - A^2 \\
&= A^2(\exp(\sigma^2) - 1)
\end{aligned}
$$

Exercise 3.18 (E/M)

$$
\begin{aligned}
f_0(p) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-P)^{\beta-1} \\
&= \frac{1}{\beta(\alpha,\beta)p^{\alpha-1}(1-P)^{\beta-1}}
\end{aligned}
$$

Data: s successful searches in a attempts

$$
f(p|(s,a)) = \frac{\begin{pmatrix} a \\ s \end{pmatrix} p^s(1-p)^{a-s} f_0(p)}{\int_0^1 \begin{pmatrix} a \\ s \end{pmatrix} p^s(1-p)^{a-s} f_0(p) dp}
$$

This is Bayes Theorem. We can simplify and write.

$$
f(p|(s,a)) = \frac{p^s(1-p)^{a-s}}{\int_0^1 p^s(1-p)^{a-s} dp}
$$

The denominator can be written in terms of the beta function.

$$
\begin{aligned}
B(s,a) &= \int_0^1 p^{s-1}(1-p)^{a-\beta-1} dp \\
B(s+1,a+1) &= \int_0^1 p^s(1-p)^{a-s} dp \\
f(p|(s,a)) &= \frac{p^s(1-p)^{a-s}}{\beta(s+1,a-s+1)}
\end{aligned}
$$

posterior probability

$$\begin{aligned}
\frac{1}{\beta(\alpha,\beta)\beta(s+1,a-s+1)} p^s p^{\alpha-1}(1-p)^{a-s}(1-p)^{\beta-1} \\
= p^{\alpha+s-1}(1-p)^{\alpha+\beta-s-1} \\
\alpha \Rightarrow \alpha+s \\
\beta \Rightarrow \alpha+\beta-s
\end{aligned}$$

The new parameters are $\alpha+s$, $\alpha+\beta-s$.